South East Asian J. of Mathematics and Mathematical Sciences
Vol. 22, No. 1 (2026), pp. 81-92

DOI: 10.56827/SEAJMMS.2026.2201.6 **ISSN (Online): 2582-0850**

ISSN (Print): 0972-7752

ON RECENT PARTITION FUNCTION OF KAUR AND RANA

Anjelin Mariya Johnson and S. N. Fathima

Department of Mathematics,
Pondicherry University,
Puducherry - 605014, INDIA

E-mail : anjelinvallialil@pondiuni.ac.in, dr.fathima.sn@pondiuni.ac.in

(**Received:** *Oct. 13, 2025* **Accepted:** *Mar. 14, 2026* **Published:** *Apr. 30, 2026)*

Abstract: Recently, Kaur and Rana introduced the partition function denoted by $\rho(n)$, where the largest part λ appears exactly once, and the remaining parts constitute a partition of λ. In this paper, we establish new generating functions for certain variants of $\rho(n)$. Further, we obtain a linear recurrence relation for our new generating function.

Keywords and Phrases: Partitions, Generating Function.

2020 Mathematics Subject Classification: 05A15, 11P83.

1. Introduction

Throughout this paper, we adopt the standard notations on partitions and q-series, as in Andrews [3] and Gasper and Rahman [7] respectively. The q-shifted factorial $(a;q)_n$ is defined by

$$(a;q)_n = \begin{cases} 1 & , \text{ for } n = 0 \\ \prod_{k=0}^{n-1} \left(1 - aq^k\right) & , \text{ for } n \geq 1, \end{cases}$$

where $(a;q)_\infty = \lim_{n\to\infty} (a;q)_n = \prod_{k=0}^{\infty} \left(1 - aq^k\right).$

Since the infinite product diverges when $a \neq 0$ and $|q| \geq 1$, whenever $(a;q)_\infty$ appears in an identity, we shall assume $|q| < 1$.

Recall that a partition of a positive integer n is a non-increasing sequence of positive integers $\lambda_1, \lambda_2, \ldots \lambda_n$, whose sum is n. Each λ_i is called a part of the partition. Let $p(n)$ denote the number of partitions of n (see [18], A000041]). The generating function for $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty},$$

with the usual convention that $p(0) = 1$. For example, the 5 partitions of 4 are

$$4,\ 3+1,\ 2+2,\ 2+1+1,\ 1+1+1+1$$

Several prominent mathematicians have contributed to the study of partitions. For a general overview of theory of partitions, we refer the reader to the monumental book of Andrews [3].

By imposing certain restrictions on the parts of the partition, one can obtain variants of the partition function. For example, a partition of n is *ℓ-regular* if none of its parts are multiples of ℓ. Let $b_\ell(n)$ denote the number of ℓ-regular partitions of n (For example, see [18], for $\ell = 2$, A000009, $\ell = 3$, A000726, $\ell = 4$, A001935 $\cdots$). The 3-regular partitions of 5 are

$$5\ ,\ 4+1\ ,\ 2+2+1\ ,\ 2+1+1+1\ ,\ 1+1+1+1+1.$$

Using elementary techniques, the generating function for $b_\ell(n)$ is given by (see [15])

$$\sum_{n=0}^{\infty} b_\ell(n)q^n = \frac{(q^\ell;q^\ell)_\infty}{(q;q)_\infty}.$$

Interestingly, in classical representation theory the number of irreducible p-modular representations of the symmetric group S_n is same as $b_p(n)$, where p is prime (see [13], [10]).

In [6], Corteel and Lovejoy introduced the overpartition function $\overline{p}(n)$, which counts the number of partitions of n wherein the first occurrence of parts may be overlined (see [18], A015128). For example, $\overline{p}(4) = 14$, since the partitions in question are

$$4\ ,\ \overline{4}\ ,\ 3+1\ ,\ 3+\overline{1}\ ,\ \overline{3}+1\ ,\ \overline{3}+\overline{1}\ ,\ 2+2\ ,\ \overline{2}+2\ ,\ 2+1+1\ ,$$

$$\overline{2}+1+1\ ,\ 2+\overline{1}+1\ ,\ \overline{2}+\overline{1}+1\ ,\ 1+1+1+1\ ,\ \overline{1}+1+1+1.$$

The generating function for $\overline{p}(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{p}(n)q^n = \frac{(q^2;q^2)_\infty}{(q;q)_\infty^2}.$$

Further, Lovejoy [14] investigated the ℓ-regular overpartition $\overline{b_\ell}(n)$, which counts the number of overpartitions of n with no parts divisible by ℓ. From the above example, it is clear that $\overline{b}_3(4) = 10$. The generating function for $\overline{b_\ell}(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{b}_\ell(n)q^n = \frac{(q^\ell;q^\ell)_\infty^2 \,(q^2;q^2)_\infty}{(q;q)_\infty^2 \,(q^{2\ell};q^{2\ell})_\infty}.$$

Similarly, the number of overpartitions of n in which only odd parts are used is denoted by $\overline{po}(n)$ (See [18], A080054), and the number of overpartitions of n in which only even parts are used is denoted by $\overline{pe}(n)$. Hence $\overline{po}(4) = 6$ and $\overline{pe}(4) = 4$. The generating functions for $\overline{po}(n)$ and $\overline{pe}(n)$ are given by(see [17], [8])

$$\sum_{n=0}^{\infty} \overline{po}(n)q^n = \frac{(q^2;q^2)_\infty^3}{(q;q)_\infty^2 (q^4;q^4)_\infty},$$

and

$$\sum_{n=0}^{\infty} \overline{pe}(n)q^n = \frac{(q^4;q^4)_\infty}{(q^2;q^2)_\infty^2},$$

respectively.

Agarwal and Andrews [1] were the first to focus on colored partition. We denote by $p_{-k}(n)$, the function which enumerates the partitions of n into k colors (See [18], $k = 2$, A005380). The generating function for $p_{-k}(n)$ (See [12]) is given by

$$\sum_{n=0}^{\infty} p_{-k}(n)q^n = \frac{1}{(q^k;q^k)_\infty}.$$

For instance, if each part of partition of 3 have colors, say red(r) and blue(b) then $p_2(3) = 10$, with the corresponding partitions

$$3_r,\ 3_b,\ 2_r + 1_r,\ 2_r + 1_b,\ 2_b + 1_r,\ 2_b + 1_b, 1_r + 1_r + 1_r,$$

$$1_r + 1_r + 1_b,\ 1_r + 1_b + 1_b,\ 1_b + 1_b + 1_b$$

In [5], Chan investigated cubic partition $a(n)$, which counts the number of partition in which the even parts can occur in two distinct colors (See [18], A002513). The generating function for $a(n)$ is given by

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{1}{(q;q)_\infty (q^2;q^2)_\infty}.$$

Recently, Hirschhorn and Sellers [9] studied the POD function, which counts the number of partitions of n wherein the odd parts are distinct (and the even parts are unrestricted) (See [18], A006950). The generating function for $pod(n)$ is given by

$$\sum_{n=0}^{\infty} pod(n)\, q^n = \frac{(q^2;q^2)_\infty}{(q;q)_\infty (q^4;q^4)_\infty}.$$

Further, Andrews, Hirschhorn and Sellers [4] studied the PED function, which counts the number of partitions of n wherein the even parts are distinct (and the odd parts are unrestricted) (See [18], A001935). The generating function for $ped(n)$ is given by

$$\sum_{n=0}^{\infty} ped(n)\, q^n = \frac{(q^4;q^4)_\infty}{(q;q)_\infty}.$$

Also, Andrews [2] studied the partition function $\mathcal{EO}(n)$ which enumerates the number of partitions of n in which every even part is less than each odd part. For example, $\mathcal{EO}(6) = 7$, with the relevant partitions being

$$6,\ 5+1,\ 4+2,\ 3+3,\ 3+1+1+1,\ 2+2+2,\ 1+1+1+1+1+1.$$

The generating function for $\mathcal{EO}(n)$ is given by

$$\sum_{n=0}^{\infty} \mathcal{EO}(n)\, q^n = \frac{1}{(1-q)(q^2;q^2)_\infty}.$$

Also, in [2] Andrews studied $\overline{\mathcal{EO}}$, the number of partitions enumerated by $\mathcal{EO}(n)$ in which only the largest even part appears an odd number of times. For example, $\overline{\mathcal{EO}}(6) = 4$, with the relevant partitions being

$$6,\ 3+3,\ 2+2+2,\ 1+1+1+1+1+1.$$

The generating function for $\overline{\mathcal{EO}}(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(n)\, q^n = \frac{(q^4;q^4)_\infty^3}{(q^2;q^2)_\infty^2}.$$

Very recently, Kaur and Rana [11] introduced the partition function $\rho(n)$ (See [18], A000065), where the largest part appears exactly once, and the remaining parts constitute a partition of that largest part. For example, $\rho(12) = 10$, and the relevant partitions are

$6+5+1$, $6+4+2$, $6+4+1+1$, $6+3+3$, $6+3+2+1$, $6+3+1+1+1$, $6+2+2+2$, $6+2+2+1+1$, $6+2+1+1+1+1$, $6+1+1+1+1+1+1$.

The generating function for the partition $\rho(n)$ is given by

$$\sum_{n=0}^{\infty} \rho(n) q^n = \frac{1}{(q^2,q^2)_\infty} - \frac{1}{(1-q^2)}. \tag{1.1}$$

In this paper, motivated by the results of Kaur and Rana, we aim to investigate generating functions for different variants of $\rho(n)$. To state our main results, we consider the following partition functions.

Definition 1.1. *For a positive integer n, we define the partition function*

- $\rho_\ell(n)$*, which counts the number of partitions of n, wherein the largest part λ appears exactly once, and the remaining parts constitute ℓ -regular partitions of λ .*
- $\overline{\rho}(n)$*, which counts the number of partitions of n, wherein the largest part λ appears exactly once, and the remaining parts constitute overpartitions of λ.*
- $\overline{\rho_o}(n)$*, which counts the number of partitions of n, wherein the largest part λ appears exactly once, and the remaining parts constitute overpartitions of λ into odd parts.*
- $\overline{\rho_e}(n)$*, which counts the number of partitions of n, wherein the largest part λ appears exactly once, and the remaining parts constitute overpartitions of λ into even parts.*
- $\overline{\rho}_\ell(n)$*, which counts the number of partitions of n, wherein the largest part λ appears exactly once, and the remaining parts constitute ℓ-regular overpartitions of λ .*

- $\rho_{pod}(n)$, *which counts the number of partitions of* n, *wherein the largest part* λ *appears exactly once, and the remaining parts constitute distinct, odd partitions of* λ.
- $\rho_{ped}(n)$, *which counts the number of partitions of* n, *wherein the largest part* λ *appears exactly once, and the remaining parts constitute distinct, even partitions of* λ.
- $\rho_{-k}(n)$, *which counts the number of partitions of* n, *wherein the largest part* λ *appears exactly once, and the remaining parts constitute* k*-coloured partition of* λ.
- $\overline{\rho}_c(n)$, *which counts the number of partitions of* n, *wherein the largest part* λ *appears exactly once, and the remaining parts constitute cubic partitions of* λ.
- $\rho_{\mathcal{EO}}(n)$, *which counts the number of partitions of* n *wherein the largest part* λ *appears exactly once and the remaining parts are partitions of* λ *where every even part is less than each odd part.*
- $\rho_{\overline{\mathcal{EO}}}(n)$, *which counts the number of partitions of* n *wherein the largest part* λ *appears exactly once and the remaining parts are partitions of* λ *enumerated by* $\mathcal{EO}(n)$ *in which only the largest even part appears an odd number of times.*

We now present our main results:

Theorem 1.1. *For* $n \geq 0$, *we have*

$$\sum_{n=0}^{\infty} \rho_{\ell}(n)q^n = \frac{(q^{2\ell};q^{2\ell})_\infty}{(q^2;q^2)_\infty} - \frac{1}{1-q^2} + \frac{q^{2\ell}}{1-q^{2\ell}}.$$

Theorem 1.2. *For* $n \geq 0$, *we have*

$$\sum_{n=0}^{\infty} \overline{\rho}(n)q^n = \frac{(q^4;q^4)_\infty}{(q^2;q^2)_\infty^2} - \frac{2}{1-q^2} + 1, \tag{1.2}$$

$$\sum_{n=0}^{\infty} \overline{\rho}_o(n)q^n = \frac{(q^4;q^4)_\infty^3}{(q^2;q^2)_\infty^2 (q^8;q^8)_\infty} - \frac{2q^2}{1-q^4} - 1, \tag{1.3}$$

$$\sum_{n=0}^{\infty} \overline{\rho}_e(n)q^n = \frac{(q^8;q^8)_\infty}{(q^4;q^4)_\infty^2} - \frac{2q^4}{1-q^4} - 1. \tag{1.4}$$

Theorem 1.3. *For $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} \overline{\rho_\ell}(n)q^n = \frac{(q^{2\ell};q^{2\ell})_\infty^2 (q^4;q^4)_\infty}{(q^2;q^2)_\infty^2 (q^{4\ell};q^{4\ell})_\infty} - \frac{2}{1-q^2} + \frac{2q^{2\ell}}{1-q^{2\ell}} + 1.$$

Theorem 1.4. *For $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} \rho_{-k}(n)q^n = \frac{1}{(q^2;q^2)^k} - \frac{kq^2}{1-q^2} - 1.$$

Theorem 1.5. *For $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} \rho_c(n)q^n = \frac{1}{(q^2;q^2)_\infty (q^4;q^4)_\infty} - \frac{2}{1-q^2} + \frac{q^6}{1-q^4} + 1 + q^2.$$

Theorem 1.6. *For $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} \rho_{pod}(n)q^n = \frac{(q^4;q^4)_\infty}{(q^2;q^2)_\infty (q^8;q^8)_\infty} - \frac{1}{1-q^2}, \tag{1.5}$$

$$\sum_{n=0}^{\infty} \rho_{ped}(n)q^n = \frac{(q^8;q^8)_\infty}{(q^2;q^2)_\infty} - \frac{1}{1-q^2}. \tag{1.6}$$

Theorem 1.7. *For $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} \rho_{\mathcal{EO}}(n)q^n = \frac{1}{1-q^2}\left[\frac{1}{(q^4;q^4)_\infty} - 1\right], \tag{1.7}$$

$$\sum_{n=0}^{\infty} \rho_{\overline{\mathcal{EO}}}(n)q^n = \frac{(q^8;q^8)_\infty^3}{(q^4;q^4)_\infty^2} - \frac{1}{1-q^4}. \tag{1.8}$$

The reminder of this paper is organized as follows. In Section 2, we focus our attention on proving our main results which are elementary in nature relying on generating function manipulations. In Section 3, we conclude this paper with an interesting recurrence relation involving partition function $\rho(n)$ which may motivate further avenues of study.

2. Proof of Theorems

We establish in this section Theorem 1.1 - 1.7.

Proof of Theorem 1.1. We have

$$\begin{aligned}\sum_{n=0}^{\infty}\rho_\ell(n)q^n &= \sum_{\substack{n=2\\ \ell|n}}^{\infty} q^n(b_\ell(n))q^n + \sum_{\substack{n=2\\ \ell\nmid n}}^{\infty} q^n(b_\ell(n)-1)q^n\\ &= \sum_{\substack{n=2\\ \ell|n}}^{\infty}(b_\ell(n))q^{2n} + \sum_{\substack{n=2\\ \ell\nmid n}}^{\infty}(b_\ell(n)-1)q^{2n}\\ &= \sum_{n=0}^{\infty}(b_\ell(n)-1)q^{2n} + \sum_{\substack{n=2\\ \ell|n}}^{\infty} q^{2n}\\ &= \sum_{n=0}^{\infty}(b_\ell(n))q^{2n} - \frac{1}{1-q^2} + \sum_{k=1}^{\infty} q^{2\ell k}.\end{aligned}$$

This completes the proof of Theorem 1.1.

Proof of Theorem 1.2. We have

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{\rho}(n)q^n &= q^2(q^{1+1}+q^{\overline{1}+1}) + q^3(q^{2+1}+q^{\overline{2}+1}+q^{2+\overline{1}}+q^{\overline{2}+\overline{1}}) + \dots\\ &= \sum_{n=2}^{\infty} q^n(\overline{p}(n)-2)q^n\\ &= 1 + \sum_{n=0}^{\infty}\overline{p}(n)q^{2n} - \frac{2}{1-q^2}.\end{aligned}$$

This settles the proof of (1.2). Finally with similar arguments we obtain (1.3) and (1.4), therefore we omit the details involved.

Proof of Theorem 1.3. We have

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{\rho_\ell}(n)q^n &= \sum_{\substack{n=2\\ \ell|n}}^{\infty} q^n\overline{b}_\ell(n)q^n + \sum_{\substack{n=2\\ \ell\nmid n}}^{\infty} q^n(\overline{b}_\ell(n)-2)q^n\\ &= \sum_{n=0}^{\infty}(\overline{b}_\ell(n)-2)q^{2n} + 2\sum_{\substack{n=2\\ \ell|n}}^{\infty} q^{2n} + 1\\ &= \sum_{n=0}^{\infty}(\overline{b}_\ell(n))q^{2n} - \frac{2}{1-q^2} + \sum_{k=1}^{\infty} q^{2\ell k} + 1.\end{aligned}$$

This completes the proof of Theorem 1.3.

Proof of Theorem 1.4. We have

$$\begin{aligned}\sum_{n=0}^{\infty}\rho_{-k}(n)q^n &= \sum_{n=2}^{\infty}q^n(p_{-k}(n)-k)q^n\\ &= \sum_{n=2}^{\infty}(p_{-k}(n)-k)q^{2n}\\ &= \sum_{n=0}^{\infty}p_{-k}(n)q^{2n}-\frac{k}{1-q^2}+q^2k-1.\end{aligned}$$

This completes the proof of Theorem 1.4.

Proof of Theorem 1.5. We have

$$\begin{aligned}\sum_{n=0}^{\infty}\rho_c(n)q^n &= q^2(q^{1+1})+q^3(q^{2_1+1}+q^{2_2+2}+q^{1+1+1})+\dots\\ &= \sum_{\substack{n=2\\ 2\mid n}}^{\infty}q^n(a(n)-2)q^n+\sum_{\substack{n=3\\ 2\nmid n}}^{\infty}q^n(a(n)-1)q^n\\ &= \sum_{n=2}^{\infty}(a(n)-2)q^{2n}+\sum_{\substack{n=3\\ 2\nmid n}}^{\infty}q^{2n}\\ &= \sum_{n=0}^{\infty}a(n)q^{2n}-\frac{2}{1-q^2}+\frac{q^6}{1-q^4}+1+q^2.\end{aligned}$$

This completes the proof of Theorem 1.5.

Proof of Theorem 1.6. We have

$$\begin{aligned}\sum_{n=0}^{\infty}\rho_{pod}(n)q^n &= q^3(q^{2+1})+q^4(q^{3+1}+q^{2+2})+q^5(q^{4+1}+q^{3+2}+q^{2+2+1})+\dots\\ &= \sum_{n=3}^{\infty}q^n(pod(n)-1)q^n\\ &= \sum_{n=0}^{\infty}pod(n)q^{2n}-\sum_{n=3}^{\infty}q^{2n}.\end{aligned}$$

This settles the proof of (1.5). Finally with similar arguments we obtain (1.6), therefore we omit the details involved.

Proof of Theorem 1.7. We have

$$\sum_{n=0}^{\infty}\rho_{\mathcal{EO}}(n)q^n = q^2(q^{1+1}) + q^3(q^{1+1+1}) + q^4(q^{3+1} + q^{2+2} + q^{1+1+1+1}) + \dots$$
$$= \sum_{n=0}^{\infty} q^n(\mathcal{EO}(n) - 1)q^n$$
$$= \sum_{n=0}^{\infty}\mathcal{EO}(n)q^{2n} - \sum_{n=0}^{\infty} q^{2n}.$$

This settles the proof of (1.7). Finally with similar arguments we obtain (1.8), therefore we omit the details involved.

3. Recurrence relation involving $\rho(n)$ and $\rho_A(n)$

In this section we recall the counting function $A(n)$ studied by Merca [16]. For a positive integer n, $A(n)$ is defined to be the sum of parts counted without multiplicity in all the partitions of n. The generating function for $A(n)$ is given by ([16], Theorem 1.2)

$$\sum_{n=1}^{\infty} A(n)q^n = \frac{1}{(q;q)_\infty}.\frac{q}{(1-q)^2}. \tag{3.1}$$

It is easy to observe for $n = 4$, we have

$$A(4) = 4 + 3 + 1 + 2 + 2 + 1 + 1 = 14.$$

We now define $\rho_A(n)$ which counts the number of partitions of n, wherein the largest part λ appears exactly once, and the remaining parts constitute the counting function $A(n)$ of λ.

We conclude with the following theorem that provides a recurrence relation involving $A(n)$ and $\rho_A(n)$.

Theorem 3.1. *For n even, we have*

$$\rho_A(n) = \frac{n}{2}\left(p(\frac{n}{2}) - 2\right) + A(n/2).$$

Proof. Thanks to (1.1) and (3.1) and induction on n we have

$$2\rho_A(n) = n(\rho(n) - 1) + 2A(n/2).$$

Employing the fact,

$$\rho(n) = p(\frac{n}{2} - 1),$$

completes the proof of Theorem 3.1.

Acknowledgements

The authors are highly grateful to the anonymous referees for their helpful suggestions that enhanced this manuscript.

References

[1] Agarwal, A. K., and Andrews, G. E., Rogers-Ramanujan identities for partitions with "N copies of N", Journal of Combinatorial Theory, Series A, 45(1) (1987), 40-49.

[2] Andrews, G. E., Integer partitions with even parts below odd parts and the mock theta functions, Annals of Combinatorics, 22 (2018), 433–445.

[3] Andrews, G. E., The Theory of partitions, Cambridge University Press, 1998.

[4] Andrews, G. E., Hirschhorn, M. D., and Sellers, J. A., Arithmetic properties of partitions with even parts distinct, Ramanujan Journal, 23 (2010), 169–181.

[5] Chan, H. C., Ramanujan's cubic continued fraction and an analog of his "most beautiful identity", International Journal of Number Theory, 6(3) (2010), 673–680.

[6] Corteel, S. and Lovejoy, J., Overpartitions, Transactions of the American Mathematical Society, 356 (2010), 1623–1635.

[7] Gasper, G., and Rahman, M., Basic Hypergeometric Series, Cambridge University Press, 2004.

[8] Hirschhorn, M. D., and Sellers, J. A., Arithmetic Properties of Overpartitions into Odd Parts, Annals of Combinatorics, 10(3) (2006), 353–367.

[9] Hirschhorn, M. D., and Sellers, J. A., Arithmetic properties of partitions with odd parts distinct, Ramanujan Journal, 22 (2010), 273–284.

[10] James, G., The Representation Theory of the Symmetric Group, Cambridge University Press, 1984.

[11] Kaur, H., and Rana, M., Partitions with unique largest part and their generating functions, (2025), arXiv preprint, arXiv:2506.11447.

[12] Keith, W. J., Restricted k-color partitions, The Ramanujan Journal, 40(1) (2016), 71-92.

[13] Kylachko, A., Modular forms and representations of symmetric groups, integral lattices, and finite linear groups, Zap. Nauchn. Sem. Leningrad Otdel, Journal of Soviet Mathematics, 26(3) (1982).

[14] Lovejoy, J., Gordons theorem for overpartitions, Journal of Combinatorial Theory A., 103(2) (2003), 393-401.

[15] Lovejoy, J., and Penniston, D., 3-regular partitions and a modular K3 surface, Contemporary Mathematics, 291 (2001), 177–182.

[16] Merca, Mircea and Yee, A. J., On the sum of parts with multiplicity at least 2 in all the partitions of n, International Journal of Number Theory, 17(3) (2021), 665-681.

[17] Munagi, A. O., and Sellers, J. A., Refining overlined parts in overpartitions via residue classes: bijections, generating functions, and congruences, Utilitas Mathematica, 95 (2014), 33–49.

[18] Sloane, N. J., The on-line encyclopedia of integer sequences, (2023), https://oeis.org.